SELECTED STANDARDS FROM THE COMMON CORE STATE STANDARDS FOR MATHEMATICS, GRADES K–3: MY REASONS FOR NOT SUPPORTING THEM

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DEY
Defending the Early Years

Foreword

Defending the Early Years (DEY) is pleased to release this new report showing how selected Common Core mathematics standards for Kindergarten-Grade 3 cannot be supported by research. The report's author, Dr. Constance Kamii, a National Advisory member of DEY, is a leading scholar and researcher studying children's understanding of mathematics.

Defending the Early Years is an organization of early childhood professionals dedicated to speaking out with well-reasoned arguments against inappropriate standards, assessments, and classroom practices. We are concerned about the rising emphasis on academic skills in early childhood classrooms today. Increasing teacher-directed instruction is leading to the erosion of play-based, experiential learning that we know children need from decades of theory and research in cognitive and developmental psychology and neuroscience.

The Common Core State Standards, standards in literacy and math for K-12 that have been adopted in more than forty states, are intensifying the academic pressures on young learners. In general, these standards do not reflect how young children learn and are not developmentally appropriate.

In January, 2015, Defending the Early Years joined with the Alliance for Childhood to release a report called *Reading Instruction in Kindergarten: Little to Gain and Much to Lose*. The report showed that the Common Core standard requiring children to read in kindergarten is not based in research. And in April of 2015, Defending the Early Years released a paper authored by renowned early childhood educator Dr. Lilian Katz called *Lively Minds: Distinctions between academic versus intellectual goals for young children*. In this paper, Dr. Katz suggests that in the early years, a major component of education must be to provide a wide range of experiences, opportunities, resources and contexts that will provoke, stimulate, and support children's innate intellectual dispositions.

Now, with this new report, Dr. Kamii shows that selected Common Core math standards for Kindergarten-Grade 3 are not grounded in the large body of research on how children learn mathematics. Dr. Kamii's approach to the teaching of mathematics is compatible with the approach described in Dr. Katz' paper, one that fosters children's in-born intellectual dispositions.

At Defending the Early Years, we are calling for removing kindergarten from the Common Core and for the convening of a task force of early childhood educators to recommend developmentally appropriate, culturally responsive guidelines for supporting young children's optimal learning birth to age 3.

Nancy Carlsson-Paige, Ed.D.
For Defending the Early Years

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Introduction

Since the first half of the 20th century, the grand theory of cognitive development has been that of Jean Piaget. True, many researchers have found fault with Piaget's ideas, but his theory still stands as the major reference point; authors almost always feel obliged to say how their work relates to that of Piaget. And Piaget's stages, from infancy through adolescence, are covered in all major child development textbooks.

Educators who have tried to use Piaget's view of developmental change have often worked under the banner of constructivism, a term that calls attention to Piaget's belief that children must "construct" their own mental structures. In the field of early childhood math education, the leading constructivist is Constance Kamii.

Kamii and other constructivists emphasize that if children are to solve problems on their own, in interaction with other children, they need time to do so. As a result, constructivist approaches can appear slower than many education officials desire. Under pressure to meet curriculum goals, teachers frequently try to accelerate learning by explaining concepts directly to their students. But the main result of this kind of instruction is that children merely acquire what Piaget called "verbalisms." They learn to repeat back the teacher's words without a true grasp of the underlying mathematical concepts. As Kamii says, children frequently acquire "social-conventional knowledge"—the correct language—but not true "logico-mathematical" understanding.

Kamii has worked for many years with early childhood teachers, experimenting with new ways of stimulating children's independent thinking. She has described many kinds of specific activities. She has also conducted systematic research to evaluate these activities and to assess how well children understand mathematical concepts. In the process, she has developed a solid sense of the kinds of mathematical concepts that children can be expected to construct at each grade.

In the essay that follows, Kamii examines the Common Core State Standards for Mathematics in grades K-3. Her essay is rigorous and academic and requires careful reading, but readers will find the effort very worthwhile. Kamii shows that many of the Common Core standards are unrealistic. Young children cannot ordinarily grasp mathematical concepts as early as the standards require. To meet the Common Core State Standards, teachers will be forced to teach ideas that sail over children's heads. Children will learn "verbalisms," memorizing statements they do not understand. They will learn to accept answers on the basis of what teachers and books say and will lose confidence in their own ability to think for themselves.

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Selected Standards from the Common Core State Standards for Mathematics, Grades K-3:
My Reasons for Not Supporting Them

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I begin by clarifying the nature of logico-mathematical knowledge, which is not mentioned anywhere in the Common Core State Standards for Mathematics (2010). Piaget (1950, 1967/1971, 1945/1951) distinguished three kinds of knowledge according to their ultimate sources: physical knowledge, social-conventional knowledge, and logico-mathematical knowledge. For him, logico-mathematical knowledge was the most important part of mathematics education.

Physical knowledge is knowledge of objects in the external world. The fact that marbles roll but blocks do not is an example of physical knowledge. The fact that a glass is likely to break if it is dropped on the floor is another example of physical knowledge. The whiteness of the paper on which these words are written is a third example. The ultimate source of physical knowledge is objects in the external world.

The ultimate source of *social-conventional knowledge* is conventions that people develop over time. An example is languages like English and Spanish. Another example is holidays like the Fourth of July. A third example is rules of etiquette.

The hardest kind of knowledge to understand is *logico-mathematical knowledge*, which consists of mental relationships. If I show you two marbles, a blue one and a red one, the reader will probably agree that the two marbles are *different*. In this situation, if I ask you if this *difference* is knowable with your eyes only, you will probably say "Yes."

Piaget would disagree with that answer. He would say that the blueness of one marble is knowable with our eyes and is physical knowledge. He would also say that the redness of the other marble is knowable with our eyes and is physical knowledge. But the *difference* between the two marbles does not exist anywhere in the observable world and is therefore not observable. The *difference* is made in the head of each individual who *thinks about* the two marbles as being different.

We can also think about the two marbles as being *similar*, and it is just as true to say that the two marbles are *similar* as it is to say that they are *different*. When we think about the marbles as being *similar*, they become similar for us at that moment, and when we think about them as being *different*, they become different for us at that moment.

A third mental relationship we can create between the same two marbles is the numerical relationship *two*. If we think about the marbles numerically as two, they become *two* for us at that moment. In other words, the ultimate source of logico-mathematical knowledge is inside each individual's head; and if *two* is logico-mathematical knowledge, all the other numbers (like "three," "ten," and "a hundred") are also logico-mathematical knowledge.

The hardest kind of knowledge to understand is logico-mathematical knowledge, which consists of mental relationships.

Number: The Synthesis of Hierarchical Inclusion and Order

Piaget (1942; Piaget & Szeminska, 1941/1965; Greco, Grize, Papert, & Piaget, 1960) went on to explain the nature of number more precisely as the synthesis of two kinds of logico-mathematical relationships: (a) hierarchical inclusion and (b) order. The presence or absence of *hierarchical inclusion* can be seen in the following experiment. If we align 5 counters and ask a 4-year-old to count them, he or she may be able to count them correctly and say that there are "five." If we then ask the child to "show me five," he or she may point to the fifth object saying "It's this one" (see Figure 1(a)). This is a child who does not yet have hierarchical inclusion. Logico-mathematically more advanced 4-year-olds show all five of the objects saying, "It's all these (see Figure 1(b))." These children mentally include "one" in "two," "two" in "three," "three" in "four," and "four" in "five" hierarchically as shown in Figure 1(b).

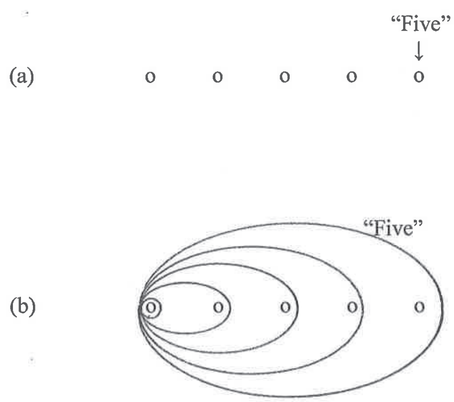

Figure 1. The meaning of "five" with or without hierarchical inclusion.

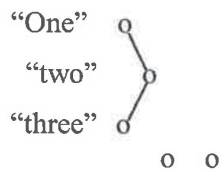

Figure 2. Ability to count with the logico-mathematical relationship of order

The absence of *order* can be seen when we ask a 4-year-old to count the objects illustrated in Figure 2. Many of them count some of the objects more than once, overlook the others, and say that there are nineteen, twenty, or forty. Logico-mathematically more advanced 4-year-olds and older children count all the objects knowing which ones have been counted, and which ones remain to be counted.

When children count the same objects more than once, many teachers correct them by showing them how to move the objects to make a different group, so that all of them will be counted, and each one will be counted only once. Kindergartners can imitate the teacher, but on the next day, they go back to their own way of counting the same objects more than once and/or overlooking the others.

Even if we do not correct children who incorrectly count objects, most of them soon become able to count them correctly. Hierarchical inclusion and order are mental relationships that children construct from within out of the network of logico-mathematical relationships shown in Figure 3. These relationships cannot be taught one by one from the outside, but teachers can indirectly encourage children to construct them by encouraging them to think in daily living and activities like Pick-Up Sticks.

	Physical knowledge	Social-conventional knowledge	Logico-mathematical knowledge				
			Logico-arith. relationships			Spatio-temporal relations.	
			Classificatory relationships	Seriational relationships	Numerical relationships	Spatial relationships	Temporal relationships
Clean-up	X	X	X	X	X	X	X
Pick-Up Sticks	X	X	X	X	X	X	X

Figure 3. The logico-mathematical framework involved in cleaning up and Pick-Up Sticks

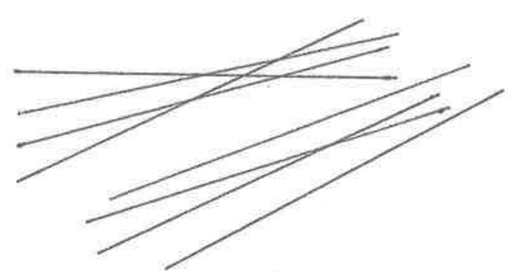

Figure 4. Eight Pick-Up Sticks that have been scattered

Hierarchical inclusion and order are mental relationships that children construct from within out of the network of logico-mathematical relationships shown in Figure 3. These relationships cannot be taught one by one from the outside, but teachers can indirectly encourage children to construct them by encouraging them to think in daily living and activities like Pick-Up Sticks.

When a child spills milk in daily living, for example, the teacher can tell him or her to "clean it up," but the teacher can also react by asking, "Would you like me to help you clean it up?" The second reaction makes the child think more than the first. Figure 3 shows the five major logico-mathematical relationships the child can make when asked, "Would you like me to help you clean it up?" *Classification* is involved when the child thinks about "cleaning it up by myself" or "cleaning it up with the teacher." *Seriation* is involved when he or she thinks about easier and harder ways of dealing with the problem. *Numerical relationships* may be involved if the child thinks about bringing one piece of paper towel, two pieces, or more. If part of the spill is on the table and part of it is on the floor, the child has to make *spatial* and *temporal relationships*. Cleaning up the floor first may result in having to clean it up again if the milk keeps dripping. (*Classificatory* and *seriational relationships* are discussed in Inhelder & Piaget (1959/1964), *numerical relationships* in Piaget & Szeminska, (1941/1965), *spatial relationships* in Piaget & Inhelder (1948/1956), and *temporal relationships* in Piaget (1946/1969).)

When children play Pick-Up Sticks, they also make many logico-mathematical relationships as shown in Figure 3. If they first look for sticks that are not touching any other stick (see Figure 4), they make *classificatory relationships* between "sticks that are touching other sticks" and "those that are not touching any other sticks." After picking up all the easiest kind to try to pick up, they look for the next easiest kind, thereby *seriating* them. *Number* is, of course, involved because the person who picks up more sticks than anybody else is the winner. When they pick up a stick that is resting on another, they make *spatial* and *temporal relationships* because they have to decide which stick to try to pick up *first*.

Number concepts are thus not constructed in a vacuum. They are built as part of a network of many logico-mathematical relationships. Other examples of activities that especially encourage children to *think* can be found in Kamii (2013, 2014a) and Kamii, Rummelsburg, and Kari (2005).

Let us examine some standards from the Common Core State Standards for Mathematics (2010). The standards cited from the CCSS appear in **bold** font below, and my reactions appear in regular font.

The standards are grouped into 11 categories called "Domains." Four of the 11 Domains appear in Grades K-3. The four are "Counting & Cardinality," "Operations & Algebraic Thinking," "Number & Operations in Base Ten," and "Measurement & Data." Each Domain is in turn subdivided into "Clusters" within which the standards appear.

Number concepts are thus not constructed in a vacuum. They are built as part of a network of many logico-mathematical relationships.

Kindergarten

A standard in the Domain of "Counting & Cardinality," in a Cluster called "Know number names and the count sequence"

"Count to 100 by ones and by tens."

I think this standard is inappropriate for kindergarten because not many 5- and 6-year-olds understand words like "forty" and "fifty." Counting is social-conventional knowledge, which is teachable, but making kindergartners count to 100 is like making them memorize nonsense syllables. Counting by tens can make even less sense to children who may or may not be able to count five objects correctly as I stated in relation to Figure 2.

Counting is social-conventional knowledge, which is teachable, but making kindergartners count to 100 is like making them memorize nonsense syllables.

Three standards in the Domain of "Counting & Cardinality," in a Cluster called "Count to tell the number of objects"

"When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object."

As I have already indicated with respect to hierarchical inclusion and order, children become able to "pair each object with one and only one number name" when they have constructed these logico-mathematical relationships. Hierarchical inclusion and order cannot be taught *directly*, but they can be taught *indirectly* by encouraging children to *think*. With the help of Figure 3, I showed that children can be encouraged to *think* in daily living while cleaning up spilled milk and in activities like Pick-Up Sticks.

Hierarchical inclusion and order cannot be taught directly, but they can be taught indirectly by encouraging children to think.

"Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted."

I do not think that "the last number name said" *tells* the child anything. *It is the child who gives meaning to words*. As I explained in relation to Figures 1(a) and 1(b), only if the child has constructed hierarchical inclusion logico-mathematically, will s/he give to the last word said the meaning that refers to the number of objects in the line.

"Understand that each successive number name refers to a quantity that is one larger."

Morf's research (Morf, 1962, cited in Kamii, 1982, p. 18) has shown that it is not until third grade that children become able to relate each subsequent number with the +1 operation. Morf came to this conclusion with experiments about "connectedness" that I replicated in the following way:

I put 15 tiny cubes in one glass and 2 cubes in another glass, and asked the child which glass had more. After the child responded that the glass containing 15 cubes had more, I quickly dropped 30 cubes one by one from a tube cut lengthwise in half into the glass that had only 2 cubes. Even kindergartners could then tell that the glass containing 32 cubes had more.

The crucial question I put to each child was, "When I was dropping one cube after another into this glass, was there a time when the two glasses had *exactly the same number*?" The percentages of children who replied, "Yes, there had to be a time when the two glasses had the same number" and gave clear, logico-mathematical justifications were distributed as shown in Table 1 (Kamii, 2014b). It can be seen in this table that it is in third grade that at least 75% of the children become able to tell logico-mathematically that *there had to be* a time when the two glasses had exactly the same number because each additional cube increased the quantity by one. In kindergarten, only 3% of the children could make such a statement.

Table 1.
Percentages in grades K-3 who demonstrated connectedness*

Kindergarten	3%
Grade 1	45%
Grade 2	66%
Grade 3	89%

*The criterion for saying that children become able to do something at a certain grade level was set at 75%.

A standard in the Domain of "Number & Operations in Base Ten," in a Cluster called "Work with numbers 11-19 to gain foundations for place value"

"Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones."

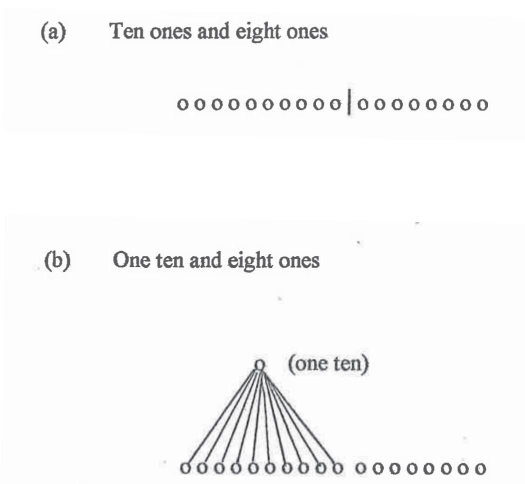

Figure 5. The difference between (a) ten ones and eight ones and (b) one ten and eight ones

This standard states that the numbers from 11 to 19 are composed of ten *ones* and some *ones*. The example they give of ten ones and eight ones (18 = 10 + 8) is illustrated in Figure 5(a).

To gain "a foundation for place value," however, children need to become able to think about a *ten* logico-mathematically as shown in Figure 5(b). Note that when children think about "one *ten*" logico-mathematically, they abstract the *ten* out of the *ones* that are in their heads and think about one *ten* and ten *ones* simultaneously.

The exercises recommended by the CCSS with objects, drawings, or equations are useless because they do not involve any logico-mathematical thinking. A better way for children to gain a foundation for place value is to be given problems like 9 + 6, which can be changed to 10 + 5. Note that when children change 9 + 6 to 10 + 5, they think about a *ten* and some *ones* simultaneously.

To gain "a foundation for place value," however, children need to become able to think about a ten logico-mathematically...

To become able to change the *ones* in 9 + 6 to a *ten* and some *ones*, children must know how to make ten. A good way for children to learn how to make ten is with games like "Tens with Nine Cards." This game requires children to find two cards that make 10. In this game, cards going up to 9 are used, and the first 9 cards are arranged randomly as shown in Figure 6. The player in this particular situation can take 3+7, 9+1, and 5+5. If s/he makes a mistake, another player will surely point it out. The player then replaces the cards taken with six cards from the deck, for the next player's turn. The person who makes more pairs than anybody else is the winner. Other games in which children find two cards that make 10 can be found in Kamii (2000).

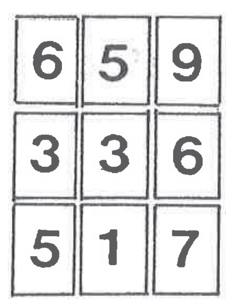

Figure 6. The arrangement of cards in Tens with Nine Cards

When children have solidly learned all the combinations of two numbers that make 10, they can be given problems like 9 + 6. Unfortunately, however, "Tens with Nine Cards" is a game that is appropriate in first grade, and kindergarten is too early for it. Likewise, a problem like 9 + 6 is appropriate early in second grade.

First Grade

A standard in the Domain of "Operations & Algebraic Thinking," in a Cluster called "Work with addition and subtraction equations"

"Determine the unknown whole number in an addition or subtraction equation relating to three whole numbers. For example, determine the unknown number that makes the equation true in each of the equations 8 + ? = 11, 5 = __ - 3, 6 + 6 = __."

Among the three examples given above, the only one that seems appropriate for first grade is "6 + 6 =__." This form seems appropriate because first graders are used to seeing it, and the sequence of numerals corresponds to their thinking about the addends first and then the total. This sequence is illustrated in Figure 7(a).

(a) The unidirectional thinking that is adequate to understand an ordinary equation

$$6 + 6 = \square$$

(b) The reversibility of thought required to understand a missing-addend problem

$$8 + \square = 11$$

Figure 7. The difference between (a) unidirectional thinking and (b) thinking that has become reversible

By contrast, when first graders see "8 + ? = 11," many of them write "19" as the answer because their thought is not yet reversible. To answer this question, children have to think about the first addend (8 in this situation), the total (11 in this situation), and come back to the second addend. This is precisely what many first graders cannot yet do, and reversibility generally develops between the ages of 7 and 8 (in second grade).

I have published an article showing that if children cannot solve missing-addend problems at the end of first grade, they become able to solve them by the end of second grade without any instruction (Kamii, Lewis, & Booker, 1998). There is therefore no need to teach missing addends in first grade.

As for "5 = __ - 3," an equation that begins with the answer is so different from the way first graders think that it cannot make any sense to them. If they put "8" as the answer after adding 5 and 3, they get the correct answer by accident.

Two standards in the Domain of "Number & Operations in Base Ten," in a Cluster called "Understand place value"

"(Understand that) 10 can be thought of as a bundle of ten ones—called a 'ten.'"

"The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones."

As stated earlier, a "ten" (shown in Figure 5(b)) is logico-mathematical knowledge, which is not observable, but a bundle of ten ones is observable. It is not possible to use this physical knowledge with the social-conventional knowledge of words like "ten" to teach the logico-mathematical knowledge of "ten".

Two standards in the Domain of "Measurement & Data," in a Cluster called "Measure lengths indirectly and by iterating length units"

"Order three objects by length; compare the lengths of two objects indirectly by using a third object."

"Compare the lengths of two objects indirectly by using a third object" refers to transitive reasoning. For example, let us say that two people (A and B) are in two different rooms, and the child is asked if a long stick can be used to know whether A is taller or B is taller. If the child has constructed transitive reasoning, he or she can use the stick to measure A's height, take the stick to the other room, and compare A's height on the stick with B's height. Children become able to coordinate these relationships in second grade according to my research (Kamii & Clark, 1997; Kamii, 2006). In other words, without any instruction, transitive reasoning (logico-mathematical knowledge) is constructed by children by second grade. I do not see why children have to be instructed in first grade to become able to do what second graders can do without any instruction.

"Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps."

The authors of the CCSS are mistaken in thinking that multiple copies of a shorter object are necessary for children to learn about unit iteration. According to the study cited above (Kamii & Clark, 1997; Kamii, 2006) and Piaget, Inhelder, & Szeminska (1948/1960), children become able to iterate a unit in fourth grade, without any instruction. To return to the example of two people (A and B) who are in different rooms, children in fourth grade become able to use a small block repeatedly without any gaps or overlaps, to know whether A is taller or B is taller. In other words, unit iteration is logico-mathematical knowledge that develops from within, out of transitivity. It is a waste of first graders' time to be taught to line up many objects and count them.

Second Grade

Three standards in the Domain of "Number & Operations in Base Ten," in a Cluster called "Use place value understanding and properties of operations to add and subtract"

"Fluently add and subtract within 100 using strategies based on place value, properties of operations and/or the relationship between addition and subtraction."

"Add up to four two-digit numbers using strategies based on place value and properties of operations."

These objectives seem acceptable for second grade because they involve two-digit numbers. However, two-digit subtraction is still too hard for some fourth graders. These standards can be accepted because they at least do not involve the social-conventional knowledge of "carrying" and "borrowing."

"Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds."

I cannot support this standard because only a minority of second graders can add and subtract three-digit numbers (Kamii, 2004). The use of "concrete models" must refer to base-ten blocks, which are useless if children have not constructed "tens" and "hundreds" in their heads. "Tens" and "hundreds" are logico-mathematical knowledge, which cannot be constructed out of physical knowledge about blocks. If the blocks seem to help, they help only to show how our system of writing works (social-conventional knowledge).

Two standards in the Domain of "Measurement & Data," and a Cluster called "Measure and estimate lengths in standard units"

"Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes."

I cannot support this objective because second graders have not constructed the logic of unit iteration in their heads (Kamii & Clark, 1997; Kamii, 2006). Second graders will become able to understand the units that are indicated on yardsticks, meter sticks, or measuring tapes when they have constructed the logic of unit iteration in fourth grade.

"Estimate lengths using units of inches, feet, centimeters, and meters."

If second graders cannot understand unit iteration, they cannot use units to estimate lengths in inches, feet, centimeters, or meters. Units to iterate are logico-mathematical knowledge, but the fact that there are 12 inches in a foot, and 100 centimeters in a meter is social-conventional knowledge.

Second graders will become able to understand the units that are indicated on yardsticks, meter sticks, or measuring tapes when they have constructed the logic of unit iteration in fourth grade.

A standard in the Domain of "Measurement & Data," in a Cluster called "Work with time and money"

"Tell and write time from analog and digital clocks to the nearest five minutes, using a.m. and p.m."

Telling time appears in a Cluster called "work with time and money," but telling time has little to do with children's thinking about time. Piaget (1946/1969) made a distinction between intuitive, preoperational time and operational time. The characteristic of operational time is that it is a deductive, logico-mathematical system. By contrast, intuitive, preoperational judgments are based on what is observable. An example of intuitive time can be seen in the following interview Piaget conducted with ROM, a 4-year-old.

ROM . . . has a small sister called Erica. How old is she? *Don't know*. Is she a baby? *No, she can walk*. Who is the older of you two? *Me*. Why? *Because I'm the bigger one*. Who will be older when she starts going to school? *Don't know* [because I don't know who will be bigger]. When you are grown up, will one of you be older than the other? *Yes*. Which one? *Don't know* [because I don't know who will be bigger]. (p. 221)

Around the age of 7 or 8, according to Piaget, children coordinate the temporal relationships they constructed before and construct operational time. When time becomes operational, children become able to deduce logico-mathematically that the age difference between two people (or two trees) always remains the same. This is what I confirmed in an experiment sketched below (Kamii & Russell, 2010).

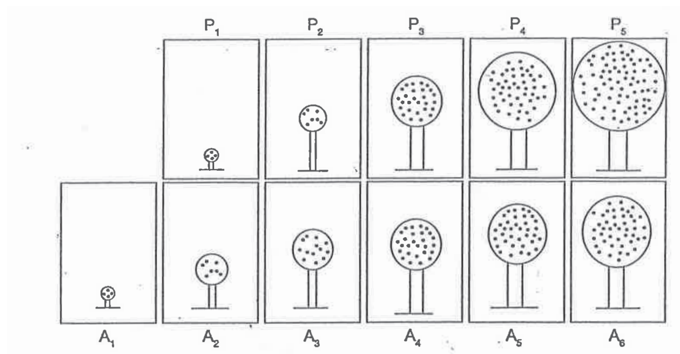

Figure 8. Eleven cards showing the growth of an apple tree and a pear tree

I placed Card A_1 (Figure 8) in front of the child saying that I planted an apple tree on its first birthday and took its picture when it was one year old. I then aligned cards A_2 through A_6, one by one in front of the child, explaining that every year on its birthday I took a picture of the apple tree that became bigger and bigger. After all six of the cards showing the apple tree were aligned, I placed the picture of a tiny pear tree (P_1) above A_2 saying that when the apple tree was 2 years old, I planted a pear tree on its birthday when it was one year old. The four other pear trees (P_2 – P_5) were then placed in a line, one by one, as I explained that the two trees had birthdays on the same day every year, and that I took a picture of each tree on the same day.

The important question I then put to each child in grades K-5 was: When I took pictures P_4 and A_5, which of the two trees was older? As can be seen in Table 2, it was in third grade that at least 75% of the children replied that the apple tree was older than the pear tree (even though the pear tree was bigger). This was about a year later than the age of 7 or 8 that Piaget reported, but the essential point was confirmed that children construct operational time.

Table 2

Judgments about Which Tree was the Older Made by Children in K-5 (in Percent)

	Kdg.	1	2	3	4	5
(n)	(31)	(27)	(29)	(29)	(33)	(35)
A_5 was older	6	24	55	79	82	94
P_4 was older	94	72	38	21	15	3

With respect to the CCSS, the point can clearly be made that being able to tell time is only social-conventional knowledge that has little to do with children's ability to think logico-mathematically about time. Schools' focus on "telling time" has little to do with children's *thinking about time*.

...being able to tell time is only social-conventional knowledge that has little to do with children's ability to think logico-mathematically about time.

Third Grade

Two standards in the Domain of "Operations & Algebraic Thinking," in a Cluster called "Represent and solve problems involving multiplication and division"

"Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. For example, describe a context in which a total number of objects can be expressed as "5 × 7."

"Interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. For example, describe a context in which a number of shares or a number of groups can be expressed as 56 ÷ 8."

These standards deal with multiplication as if it could easily be taught. However, multiplication is much more complicated than repeated addition because it requires thinking at two hierarchical levels. For example, in repeated addition like 4 + 4 + 4, all the 4s are ones as can be seen in Figure 9(a). In 3 × 4, by contrast, the 4s are still ones, but the 3 is not the same kind of number as the 4s. As shown in Figure 9(b), the "3" in "3 × 4" is a higher-order number that means "3 groups (of 4)." This hierarchical thinking was not possible for about a third of the middle-class third graders interviewed by Clark and Kamii (1996).

...multiplication is much more complicated than repeated addition because it requires thinking at two hierarchical levels.

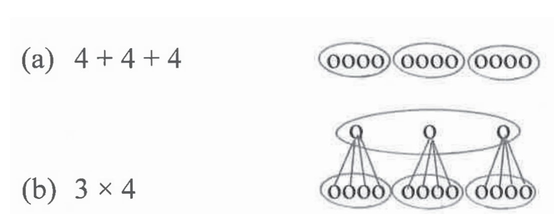

Figure 9. The difference between (a) repeated addition (4+4+4) and (b) multiplication (3 x 4)

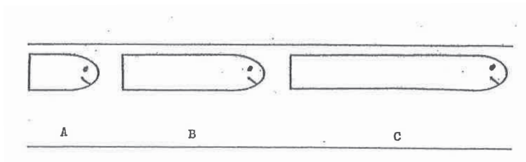

Figure 10. The fish (eels) used in the multiplicative-thinking task

In individual interviews, Clark and Kamii presented each child with three cardboard fish (see Figure 10). Fish A, B, and C were 5, 10, and 15 cm long, respectively, and the child was told, "This fish (B) eats 2 times what this fish (A) eats, and this big fish (C) eats 3 times what the little one (A) eats. This fish (B) eats 2 times what this fish (A) eats because it is 2 times as big as this one (A)." The interviewer demonstrated by showing that A could be placed on B two times, and it could be placed on C three times.

The child was then given about 50 counters and asked the following five questions:

Table 3
Number and Percentage of Children at Each Developmental Level by Grade

	Grade				
	1	2	3	4	5
Level					
1[a]	8 (13.8)	1 (1.5)	--	--	--
2[b]	31 (53.4)	28 (43.1)	8 (13.6)	12 (15.4)	5 (6.6)
3[b]	8 (13.8)	7 (10.8)	13 (22.0)	2 (2.6)	2 (2.6)
4A[c]	10 (17.2)	23 (35.4)	25 (42.4)	42 (53.8)	32 (42.1)
4B[c]	1 (1.7)	6 (9.2)	13 (22.0)	22 (28.2)	37 (48.7)
	58 (99.9)	65 (100)	59 (100)	78 (100)	76 (100)

Note: Values enclosed in parentheses represent percentages.
[a]Below additive level
[b]Additive level
[c]Multiplicative level

a. If A receives 2 counters, how many should B and C get?
b. If B receives 4 counters, how many should A and C get?
c. If C receives 9 counters, how many should A and B get?
d. If A receives 5 counters, how many should B and C get?
e. If A receives 7 counters, how many should B and C get?

If a child answered a question incorrectly, "c" for example, a counter-suggestion was offered: "Another boy/girl told me that if this big fish (C) gets 9 counters, the little fish (A) should get 3 because 9 (pointing to the 9 counters rearranged into three groups of 3) is 3 times what this is (pointing to the 3 counters given to A). And this fish (B) should get 6 because (pointing to 6 counters arranged into two groups of 3) 6 is 2 times what this is (pointing to the 3 counters given to A). What do you think of his/her idea?" After the child gave an opinion, the interviewer always asked for an explanation.

Table 3 shows the five levels found with 336 children in grades 1-5. Multiplicative thinking was demonstrated by children at Levels 4A and 4B. It can be seen in this table that only 22% of the third graders immediately made multiplicative relationships (Level 4B), and the percentages increased only to 28.2% in fourth grade and to 48.7% in fifth grade. Most of the third and fourth graders (42.4% and 53.8%, respectively) made multiplicative relationships only after a counter-suggestion (Level 4A).

The children at Levels 2 and 3 exhibited additive thinking even after the counter-suggestion. If A received 4, Level-2 children typically gave 5 counters to B (because 4+1=5) and 6 counters to C (because 5+1=6). A typical example of Level 3 was to give 6 counters to B (because 4+2=6) and 7 counters to C (because 4+3=7).

The "fish" task is useful for two reasons for identifying children who cannot yet think multiplicatively:

a. It involves small and easy multipliers like "2 times" and "3 times."
b. It lets the child show multiplicative thinking with actions (by making 2 groups of 2, and 3 groups of 2, for example) without having to give precise numbers (such as 4 for "2 x 2," and 6 for "3 x 2").

With respect to the CCSS, it can be concluded that third grade is too early to expect all students to be able to multiply. It can be seen in Table 3 that 35.6% of our third graders were at Levels 2 and 3 (additive thinking).

...third grade is too early to expect all students to be able to multiply.

A standard in the Domain of "Operations & Algebraic Thinking," in a Cluster called "Multiply and divide within 100"

"Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that 8 × 5 = 40, one knows 40 ÷ 5 = 8) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers."

This standard essentially states that by the end of third grade everybody must know all the multiplication tables by heart. If 35.6% of our middle-class third graders do not even understand what "2 times" and "3 times" mean, this memorization amounts to the memorization of nonsense syllables for at least a third of the students.

A standard in the Domain of "Number & Operations in Base Ten," in a Cluster called "Use place value understanding & properties of operations to perform multi-digit arithmetic"

"Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction."

This is the first time that the CCSS requires students to use the algorithms (rules) of "carrying" and "borrowing." I think this is too early, as I have seen too many third graders unlearn place value as a result of using these algorithms (rules).

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When children use their own invented procedures, they always proceed from the big units to the small units. To do 87 + 24, for example, they invariably do 80 + 20 = 100, 7 + 4 = 11, and 100 + 11 = 111. To do 26 – 17, they do 20 – 10 = 10 first and proceed in three different ways. One is to do 10 + 6 first and then 16 – 7 = 9. The second way is to do 10 – 7 first and then 3 + 6 = 9. The third way is to do 6 – 7 = "1 less than zero" and subtract 1 from 10. (These procedures can be seen in a videotape entitled "Double-Column Addition," which is available on my website [Constancekamii.org] in a category called "Videos Related to Math Education.")

As stated in Kamii and Dominick (1998) with evidence, the teaching of "carrying" and "borrowing" is harmful to children for two reasons:

a. They make children give up their own way of thinking.

> Children's own way is to go from the big unit to the small unit, but the algorithms make them go from the small unit to the big unit. Since children cannot find any compromise between the two ways, the only way they can obey the teacher is by giving up their own way of thinking. When we give up our own way of thinking logically, we all become less intelligent.

b. Algorithms make children unlearn place value.

When children are given the following example,

$$\begin{array}{r} 456 \\ \underline{+\ 789} \end{array},$$

they naturally say and think, "Four hundred and seven hundred is one thousand and one hundred (400 + 700 = 1,100). Fifty and eighty is a hundred and thirty (50 + 80 = 130). Six and nine is fifteen, and 1,100 + 130 + 15 = 1,245." They thus strengthen their knowledge of place value by using it.

When they use "carrying," by contrast, children unlearn place value in the following way. They say and think "Six and nine is fifteen, put down the five, and carry the one." They then say and think, "One and five and eight is fourteen. Put down the four, and carry the one . . . ."

"Carrying" is useful for people who already know that the "1" and "5" and "8" in the above problem mean "ten," "fifty," and "eighty." For children who are not sure about place value, however, "carrying" serves to "unteach" place value.

As stated in Kamii and Dominick (1998) with evidence, the teaching of "carrying" and "borrowing" is harmful to children for two reasons:
a. They make children give up their own way of thinking.
b. Algorithms make children unlearn place value.

A standard in the Domain of "Measurement & Data," in a Cluster called "Geometric measurement: understand concepts of area and relate area to multiplication and to addition"

"Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths."

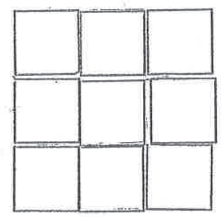

Figure 11. A 3 x 3 arrangement of tiles

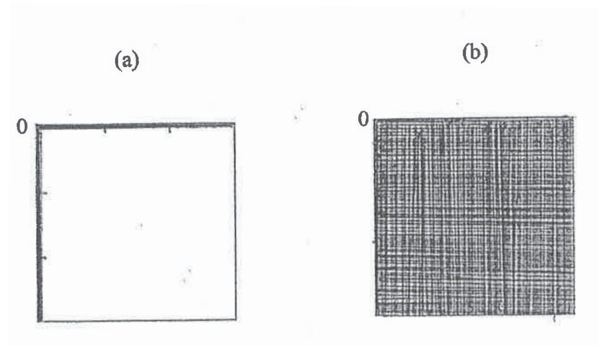

Figure 12. Two lengths perpendicular to each other (a) before and (b) after the construction of infinitesimally close parallel lines

The CCSS argues as if tiling naturally developed into the formula of length × width. However, Piaget, Inhelder, and Szeminska (1948/1960) stated that it is easy for 9- and 10-year-olds to know that there are 9 tiles in Figure 11 because tiles are discrete, and children can easily multiply 3 by 3. The authors went on to point out that lines and areas are continuous quantities, and 9- and 10-year-olds cannot understand how lines produce areas. Given two lengths of 3 cm each, perpendicular to one another and starting from the same point of origin as shown in Figure 12(a), these children cannot understand what the two lines have to do with an area.

Starting at the age of 11 or 12, when formal operations begin to develop, adolescents begin to understand that the area of a square is given by the length of its sides. "But such statement is intelligible only if it is understood that the area itself is reducible to lines, because a two-dimensional continuum amounts to an uninterrupted matrix of one-dimensional continua (Piaget, Inhelder, & Szeminska, 1948/1960, p. 350).” Such a matrix of infinitesimally close parallel lines is illustrated in Figure 12(b).

Starting at the age of 11 or 12, when formal operations begin to develop, adolescents begin to understand that the area of a square is given by the length of its sides.

The National Assessment of Educational Progress (NAEP) has been showing repeatedly that even 7th graders or 13-year-olds cannot use the formula of length × width. When presented with a 4 cm by 6 cm rectangle (shown in Figure 13), for example, only half of the students gave the correct answer. The percentage answering the question correctly was 51 in the Second Assessment, and 48 in the Third Assessment (Lindquist, Carpenter, Silver, & Matthews, 1983). In the Fourth Assessment (Lindquist & Kouba, 1989), when the rectangle was 5 × 6, the percentage was 46.

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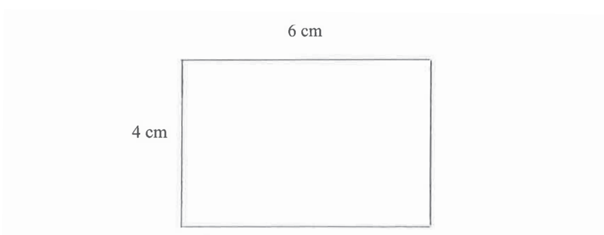

Figure 13. A rectangle shown in the Second and Third National Assessment of Educational progress with the question “What is the area of this rectangle?”

Kamii and Kysh (2006) made a 3 × 3 square and a 2 × 4 rectangle on two geoboards and asked 4th, 6th, 8th, and 9th graders (a) which geoboard would have more to eat if they showed chocolate bars, and (b) to count whatever they needed to count to prove the correctness of their answers. As can be seen in Table 4, only those in the advanced sections of 8th and 9th grades counted squares and said that the 3 × 3 square would have more to eat. Most of the others counted pegs to prove the correctness of their answers!

Table 4

*Percentages of Children Who Counted Squares and Pegs**

	Grade level					
	4	6	8		9	
			Reg.	Adv.	Reg.	Adv.
	(38)	(65)	(27)	(23)	(30)	(27)
Counted squares (9 vs. 8)	16	56	41	83	53	93
Counted pegs (16 vs. 15)	68	38	59	17	47	7

*The numbers in parentheses indicate the number in each sample.

Wilder (2014) conducted a similar study, but he drew a rectangle on each of two sheets of paper and asked students which rectangle would have more to eat if they were chocolate bars. The rectangles were 7” × 3” and 5” × 4”, respectively. He offered to students in grades 4, 6, and 8 the following four tools, one by one, asking them if it could be used to prove that the bar they chose indeed had more to eat.

a. A 1-inch stick (which could be iterated to measure the length of each side)
b. A ruler (which could be used to measure the length of each side)
c. A tile (which could be used to iterate to measure the area of each rectangle)
d. Additional tiles (8 to 21, which could be used to tile each rectangle)

If a child could use the 1-inch stick satisfactorily, the interview was ended. If not, the interviewer went on to the next question with a ruler. If neither the 1” stick nor the ruler could be used satisfactorily, the interviewer went on to the use of a tile (question c); and if a tile could not be used, 8 to 21 were added (question d).

As can be seen in Table 5, the percentages in grades 4, 6, and 8 who used length × width by using either the 1" stick or the ruler were only 5, 15, and 25%, respectively. The percentages who used 1-21 tiles were 50, 75, and 70, respectively. The other students did not even try to measure the area; they measured other variables such as the perimeter. Wilder concluded that length × width was too hard even for eighth graders to remember to use. Because Wilder's eighth graders did not include an "advanced" group, only 25% came out looking capable of using the formula of length × width.

Table 5

Percentages in Grades 4, 6, and 8 Who Quantified Area Correctly

Grade	n	With 8-21 tiles	With 1 tile	With ruler	With 1-inch stick
4	20	40	10	5	0
6	20	45	30	10	5
8	20	50	20	10	15

Conclusion

I tried to show in the foregoing discussion that children in grades K-3 cannot be expected to meet the standards specified by the CCSS because they are set at grade levels that are too early. These high expectations can be detrimental.

...children in grades K-3 cannot be expected to meet the standards specified by the CCSS because they are set at grade levels that are too early.

Another point stressed in the discussion is that most of the standards set by the CCSS involve logico-mathematical knowledge which is not teachable. Because the authors of the CCSS are not aware of the difference between logico-mathematical knowledge and social-conventional knowledge, they urge the direct teaching of logico-mathematical knowledge.

A question I would like to pose before concluding is: Why did the authors of the CCSS not consider the large body of data available from research? It is obvious to any teacher of children in grades K-3 that the standards discussed above are too hard for most children. Ravitch (2014) said, "The makeup of the work group (who wrote the Standards) helps to explain why so many people in the field of early childhood education find the CCSS to be developmentally inappropriate. There was literally no one on the writing committee (with one possible exception) with any knowledge of how very young children learn."

Because the authors of the CCSS are not aware of the difference between logico-mathematical knowledge and social-conventional knowledge, they urge the direct teaching of logico-mathematical knowledge.

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Appendix: Examples of How Constructivist Teachers Teach Math

Videotapes

"First Graders Dividing 62 by 5"
Available from Teachers College Press (www.tcpress.com) and amazon.com

"Double-Column Addition"
"Multidigit Division"
Available on my website (Constancekamii.org) in a category called "Videos Related to Math Educ."

Articles

"Lining Up the 5s" (a card game for kindergarten)
"Board Game Sorry!" (a board game for kindergarten and first grade)
Available on my website (Constancekamii.org) in a category called "Articles Available for Downloading"

Books published by Teachers College Press

C. Kamii, Young Children Reinvent Arithmetic (about 1st grade), 2nd ed. (2000).

C. Kamii, Young Children Continue to Reinvent Arithmetic, 2nd Grade, 2nd ed. (2004).

C. Kamii, Young Children Continue to Reinvent Arithmetic, 3rd Grade. (1994).

Constance Kamii, Ph.D., studied under Jean Piaget for 15 years on a half-time basis. Since 1980, she has been working with teachers to develop ways of using his theory to teach arithmetic in classrooms. The findings are described in *Young Children Reinvent Arithmetic (about first grade); Young Children Continue to Reinvent Arithmetic, 2nd Grade; and Young Children Continue to Reinvent Arithmetic, 3rd Grade.*

Defending the Early Years (DEY) was founded in 2012 to rally educators to take action on policies that affect the education of young children. DEY is committed to promoting appropriate practices in early childhood classrooms and supporting educators in counteracting current reforms which undermine these appropriate practices. DEY is a non-profit project of the Survival Education Fund, Inc., a 501 (c) 3 educational organization.

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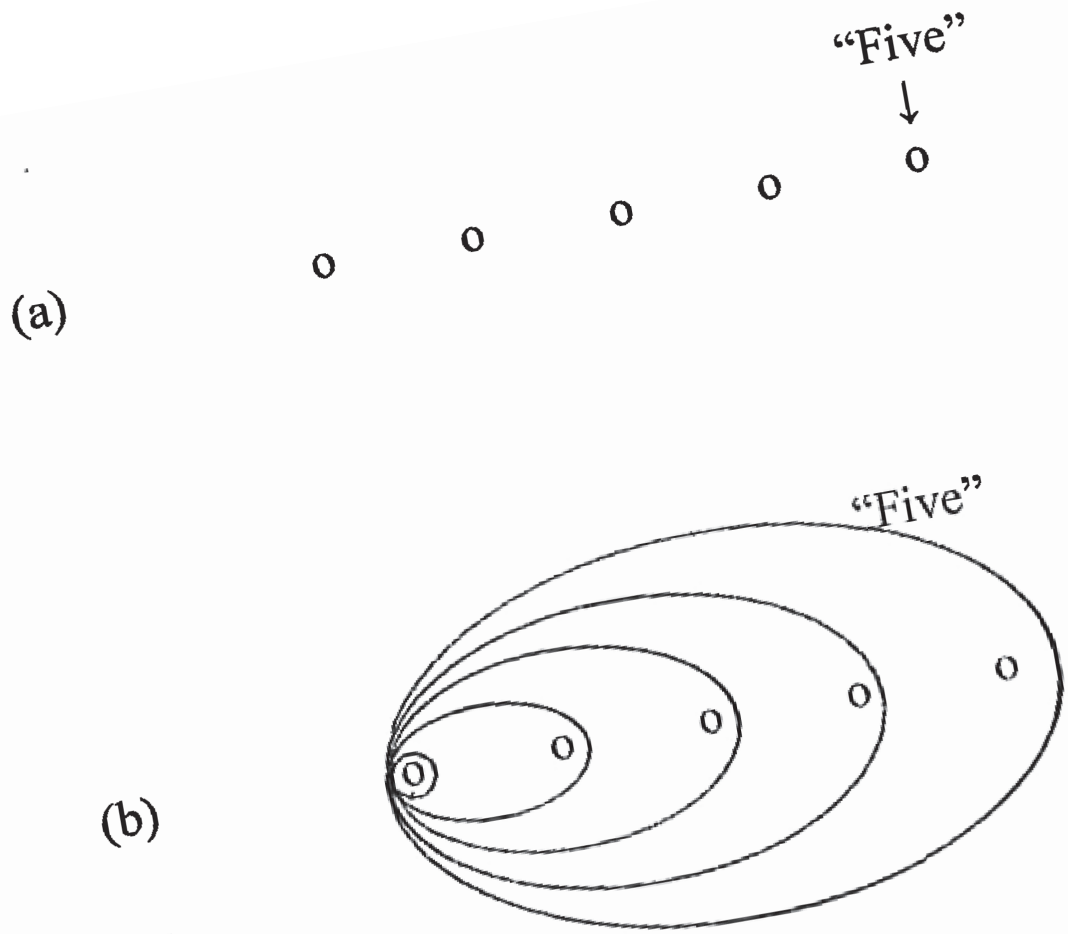
SELECTED STANDARDS FROM THE COMMON CORE STATE STANDARDS FOR MATHEMATICS, GRADES K–3: MY REASONS FOR NOT SUPPORTING THEM

CONSTANCE KAMII
THE UNIVERSITY OF ALABAMA AT BIRMINGHAM



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